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Compact Star Model Through Hypergeometric Differential Equation

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Abstract

In this analysis, for a static, spherically symmetric, anisotropic and charged distribution of matter, we present a new class of exact solutions to the Einstein-Maxwell system. This is obtained by assuming specific forms for one of the gravitational potentials, electric field intensity and measure of anisotropy. We modify the condition of pressure isotropy to a second order differential equation which is the master equation of the entire system by introducing new metric functions. This master equation of the Einstein-Maxwell system is reduced to a hypergeometric differential equation with the assistance of a transformation. It is then possible to find exact solutions which can be written explicitly in terms of elementary functions for specific values of model parameters involved. Under certain choices of the model parameters, the new class of solutions is shown to regain some of the previously reported realistic models of compact stars. We illustrate that it is possible to express our class of solutions in a simple closed form so as to examine its physical viability for the studies of relativistic compact stars.

Keywords: Einstein-Maxwell system, compact stars

2. Introduction

This analysis is to construct new physical solutions to the Einstein-Maxwell system which satisfy the physical criteria: the gravitational potentials, electric field intensity, charge distribution and matter distribution should be well-behaved and regular throughout the star. A number of general methods have been proposed to solve the Einstein-Maxwell system for static gravitational fields. Thirukkanesh and Maharaj derived a new class of solution in closed form using a symmetric series analysis. This approach produces a number of difference equation which must be solved explicitly from first principles. Solutions in terms of elementary functions can be retrieved by placing restriction on the parameters. In this analysis we reduce the solutions of the field equations to a hypergeometric equation. Our results show that particular solutions, both with charged and uncharged matter, may be extracted from the hypergeometric equation. Key advantages of our method compared to the series method are highlighted and we depict the restrictions on solutions allowed on physical grounds.

In order to integrate the field equations, several restrictions have been imposed by investigators on the geometry of space time and the matter content. Two key procedures have been adopted to solve these equations for spherically symmetric and static manifolds. At first, the coupled differential equations are solved by computation after choosing an equation of state. As the second point, the exact Einstein-Maxwell solutions can be derived by specifying the geometry and the forms for electric field and anisotropic factor. We adopt the latter technique in an attempt to find solutions in terms of hypergeometric functions that are suitable for the description of relativistic charged stars with anisotropic pressure.

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3. Methodology

3.1 Field Equations

The space-time is spherically symmetric and static which is consistent with the study of charged compact objects. Thus, there are coordinates (t, r, θ, ϕ) such that the line element is of the form

$$ds^2 = -e^{2(r)}\,dt^2 + e^{2\lambda(r)}\,dr^2 + r^2(d\theta^2 + sin^2\theta d\phi^2) \qquad (1)$$

Where (r) and (r) are arbitrary functions in the radial coordinate r. For an anisotropic perfect fluid, the Einstein-Maxwell field equation can be written in the form

$$\frac{1}{r^2}\,[(1 - e^{-2\lambda})]' = \rho + \frac{1}{2}E^2 \qquad (2a)$$

$$\frac{-1}{r^2}(1 - e^{-2\lambda}) + \frac{2\,\nu'}{r}e^{-2\lambda} = p_r - \frac{1}{2}\,E^2 \qquad (2b)$$

$$e^{-2\lambda}\,(\nu'' + \nu'^2 + \frac{\nu\prime}{r} - \nu'\lambda' - \frac{\lambda'}{r}) = p_t + \frac{1}{2}E^2 \qquad (2c)$$

$$\sigma = \frac{1}{r^2}e^{-}(r^2E\,)' \qquad (2d)$$

- For line element (1) where ρ, p_r, p_t, E and σ are energy density, radial pressure, tangential pressure, the electric field and proper charge density and primes denote the differentiation with respect to the variable *r*.

- A different but equivalent form of the Einstein-Maxwell field equations is obtained if we introduce the transformation:

  $$A^2y^2(x) = e^{2\nu(r)},\ Z(x) = e^{-2\lambda(r)},\ x = Cr^2 \qquad (3)$$

  where *A* and *C* are constants

Under the transformation (3), the system (2) becomes

$$\frac{1-Z}{x} - 2\dot{Z} = \frac{\rho}{c} + \frac{E^2}{2c} \qquad (4b)$$

$$4Z\,\frac{\dot{y}}{y} + \frac{Z-1}{x} = \frac{p_r}{c} - \frac{E^2}{2c} \qquad (4b)$$

$$4Zx^2\ddot{y} + 2\dot{Z}x^2\dot{y} + (\dot{Z}x - Z + 1 - \frac{\Delta x}{c} - \frac{E^2x}{c})y = 0 \qquad (4c)$$

$$\frac{\sigma^2}{c} = \frac{4Z}{x}\,(x\dot{E} + E)^2 \qquad (4d)$$

The quantity $\Delta = p_t - p_r$ is called the measure of anisotropy.

As derived above, we obtain a nonlinear system consists of four independent equations in the seven unknowns ρ, p_t, p_r, E, y, Z and σ. Thus we have the freedom to choose any three variables.

3.2 Choice of potential

We solve the Einstein-Maxwell system (4) by making explicit choice for Z

$$Z = \frac{1+kx}{1+x} \tag{5}$$

Where k is a real constant. In (5) we take k≠1 to avoid negative energy density. On substituting (5) into (4c) we obtain

$$4(1+kx)(1+x)\ddot{y} + 2(k-1)\dot{y} + \left(1-k-\frac{E^2(1+x)^2}{cx}-\frac{\Delta(1+x)^2}{cx}\right) y = 0 \tag{6}$$

It is convenient to introduce a new independent variable X and relevant transformation is given by

$$1+x = KX,\ K = \frac{k-1}{k} \text{ and } Y(x) = y(x)$$

Then equation (6) can be written as

$$X(1-X)\frac{d^2Y}{dX^2} - \frac{1}{2}\frac{dY}{dX} + \left(\frac{K}{4} + \frac{K^2(1-K)E^2X^2}{c(KX-1)} + \frac{K^2(1-K)\Delta X^2}{c(KX-1)}\right)Y=0 \tag{7}$$

We observe that (7) is simplified if we make the choice

$$E^2 = \frac{\propto}{4}\frac{c(KX-1)}{K^2(1-K)X^2} \quad \text{and} \quad \Delta = \frac{\beta}{4}\frac{c(KX-1)}{K^2(1-K)X^2}$$

Where α and β are constants.

The electric field intensity E and Δ vanishes at the center of the star, and remains continuous and bounded in the interior of the star for a wide range of values of the parameter K. Thus this choices of E and Δ are physically reasonable.

On substituting the above choices in (7), we obtain

$$X(1-X)\frac{d^2Y}{dX^2} - \frac{1}{2}\frac{dY}{dX} + \left(\frac{K}{4} + \frac{\propto}{4} + \frac{\beta}{4}\right)Y=0 \tag{8}$$

Note that (8) is a special case of the hypergeometric differential equation.

3.3 Known Cases

A variety of new solutions, in terms of elementary and special functions, can be obtained from (8) for particular values of α, β and K.

- As a first example we take $\alpha=0$, $\beta=0$ and K= -1 (k=½)

Then (8) admits the two linearly independent hypergeometric functions

$$Y_1 = F(-½,-½: -½:X) \quad Y_2 = X^{3/2} F(1,1:5/2:X)$$

It is possible to express these solutions in terms of elementary functions and we get

$y_1(x) = (2 + x)^{1/2}$ and $y_2(x) = (2 + x)^{1/2}\ln[(1 + x)^{1/2}+ (2 + x)^{1/2}]^2$-2 $(1 + x)^{1/2}$

Then the general solution is

$y(x) = C_1 (2 + x)^{1/2} + C_2 [(2 + x)^{1/2} \ln[(1 + x)^{1/2}+ (2 + x)^{1/2}]^2$-2 $(1 + x)^{1/2}]$

This solution was found previously by John Maharaj (2011).

- As a second example we take α=0, β=0 and K=3 (k=-½)

In this case the two linearly independent hypergeometric functions are

Y_1 =F (½, -3/2 : -½ :X) and $Y_2 = X^{3/2}$ F(2,0:5/2:X)

These quantities are equivalent to the elementary functions

$y_1(x) = (2 - x)^{1/2}$ (2x+5)

$y_2(x) = (1 + x)^{3/2}$

Then the general solution is

$y(x) = C_1(2 - x)^{1/2}$ (2x+5) $+C_2(1 + x)^{3/2}$

Which correspond to the neutron star model of Durgapal and Bannerji model (1983).

4. Results and Discussion

If we choose the parameter values α=⅓, β=⅔ and K=-2(k=⅓)

Then (8) admits the two linearly independent hypergeometric functions

Y_1 = F (-½,-½: -½:X) $Y_2= X^{-3/2}$ F(1,1:5/2:X)

These quantities are equivalent to the elementary functions

$y_1(x)= (2 + x)^{1/2}$

$y_2(x)= 2(2 + x)^{1/2} \ln [(1 + x)^{1/2}+ (2 + x)^{1/2}]$-2 $(1 + x)^{1/2}$

We can rewrite the general solution as

$Y(x) = (2 + x)^{1/2} (C_1 + 2C_2 \ln [(1 + x)^{1/2}+ (2 + x)^{1/2}])$- $2C_2 (1 + x)^{1/2}$

Then the complete solution to the Einstein-Maxwell system can be expressed as

$$e^{2\lambda} = \frac{3(1+x)}{3+x}$$

$$e^{2\nu} = A^2((2 + x)^{1/2}(C_1 + 2C_2 \ln [(1 + x)^{1/2}+ (2 + x)^{1/2}])\text{-}2C_2 (1 + x)^{1/2})^2$$

$$\frac{E^2}{c} = \frac{x}{36(1+x)^2}$$

$$\frac{\rho}{c} = \frac{(144+47x)}{72(1+x)^2}$$

.

$$\frac{\sigma^2}{c} = \frac{(x+3)^3}{108(1+x)^5}$$

$$\frac{p_r}{c} = -\frac{(47x+48)}{72(1+x)^2} + \frac{4}{3}\frac{(3+x)}{(1+x)} \times \frac{C_1+2C_2H}{2C_1(2+x)+4C_2((2+x)H-\sqrt{(1+x)(2+x)})}$$

$$\frac{p_t}{c} = -\frac{(43x+48)}{72(1+x)^2} + \frac{4}{3}\frac{(3+x)}{(1+x)} \times \frac{C_1+2C_2H}{2C_1(2+x)+4C_2((2+x)H-\sqrt{(1+x)(2+x)})}$$

$$\frac{\Delta}{c} = \frac{x}{18}(1+x)^2$$

Where H = $\ln(\sqrt{1+x}+\sqrt{2+x})$

The gravitational potentials v and λ are well behaved and continuous in the interior of the star. This is also true for the energy density ρ, the pressure p, the electric field intensity E and the charge density σ. These quantities remain finite and non-singular.

Therefore, this new solution to the Einstein-Maxwell systems presented are physically reasonable.

5. Conclusion

The aim of this analysis is to find new exact solutions to the Einstein-Maxwell systems with a barotropic equation of state for static spherically symmetric gravitational fields. Solutions of the field equation were obtained after specifying one of the potentials, electric field and the anisotropic factor. We regained models found previously. Also we have demonstrated an exact model which is physically reasonable.

6. References

1. Komathiraj K and Maharaj S D, Classes of exact Einstein-Maxwell solutions, Gen.Relativ.Gravit.39,2079 (2007).
2. Maharaj S D and Thirukkanesh S, Some new static charged spheres, Nonlinear Analysis: Real World Applications 10, 3396 (2009).
3. Thirukkanesh S and Maharaj S D, Charged anisotropic matter with a linear equation of state, Class.Quantum Grav. 25,235001 (2008).
4. Hansraj S and Maharaj S D, Charged analogue of Finch-Skea stars, Int. J.Mod.Phys. D 15, 1311 (2006).
5. Durgapal M C and Banerji R, New analytical stellar model in general relativity, Phys.Rev.27, 328 (1983).
6. John A H and Maharaj S D, Relativistic stellar models, prammana J. phys, 77,461(2011).